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Representations of Algebras

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PREFACE

During the recent years, a number of significant advances have been made in the theory of representations of algebras. Therefore, a meeting reflecting these advances and exploring the relationship among the results in this area was desirable. Such a meeting, the International Conference on Representations of Algebras was held at Carleton University, Ottawa, on September 3 - 7, 1974. It is our pleasure to acknowledge with gratitude the financial assistance of the National Research Council of Canada to support the Conference.

The program of the conference included 18 invited addresses and 10 contributed papers. In accordance with the Springer Lecture Notes policy, the papers outside the scope of the Conference and abstracts do not appear in this volume; on the other hand, the papers of P. Gabriel, M. Loupias, L. A. Nazarova and A. V. Roiter - M. M. Kleiner, who were unable to attend the meeting, are included. The papers appear in the form submitted by the authors; only very few technical alterations have been made.

We wish to thank Carleton University for the support in organizing the Conference. In particular, we wish to express our sincere thanks to the Secretary of the Conference, Luis Ribes, for his efficiency and success in making the meeting run smoothly and to Donna Desaulniers and Susanne Greening for their most appreciated secretarial assistance.

Oberwolfach, May 1975 Vlastimil Dlab and Peter Gabriel

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ALMOST SPLIT SEQUENCES I

Maurice Auslander

The main purpose of these talks is to introduce the notion of almost split sequences. The first talk is devoted to giving various consequences of their existence in order to indicate the diversity of their applicability. The second talk is devoted to a more detailed, but by no means definitive, examination of these sequences themselves. This is based on the expectation that almost split sequences will prove to be a useful invariant for studying indecomposable modules. Proofs for these results will appear elsewhere (see [1] and [2] for example).

Throughout this discussion all our rings are artin algebras. We recall that a ring Λ is said to be an artin algebra if it is finitely generated as a module over its center C and C is a commutative artin ring. A ring Λ is an artin algebra if and only if Λ has the structure of an R-algebra with R a commutative artin ring which has the additional property that Λ is a finitely generated R-module.

Clearly every artin algebra is a two-sided artin ring (the converse is not true). We now list some of the properties of artin algebras that we shall need which do not hold for arbitrary two-sided artin rings.

Suppose M is a finitely generated Λ-module. Then $\mathrm{End}_\Lambda(M)$, the endomorphism ring of M, is an artin algebra and hence a two-sided artin ring. Moreover an injective envelope $I(M)$ of M is a finitely generated Λ-module.

Let C be the center of Λ. If we denote by I the injective envelope over C of $C/\mathrm{rad}\, C$, then $\mathrm{Hom}_C(M,I)$ is a finitely generated right Λ-module or, equivalently, a finitely generated Λ^{op}-module where Λ^{op} is the opposite ring of Λ. If we denote by $\mathrm{mod}\,\Lambda$ and $\mathrm{mod}\,\Lambda^{op}$, the category of finitely generated Λ-modules, then we obtain the well known duality $D\colon \mathrm{mod}\,\Lambda \longrightarrow \mathrm{mod}\,\Lambda^{op}$ given by $D(X) = \mathrm{Hom}_C(X,I)$ for all X in $\mathrm{mod}\,\Lambda$ (all X in $\mathrm{mod}\,\Lambda^{op}$).

Associated with category $\mathrm{mod}\,\Lambda$ are the two important additive categories $\underline{\mathrm{mod}}\,\Lambda$ and $\overline{\mathrm{mod}}\,\Lambda$. The objects of $\underline{\mathrm{mod}}\,\Lambda$ ($\overline{\mathrm{mod}}\,\Lambda$) are the same as those of $\mathrm{mod}\,\Lambda$ but the morphisms from A to B in $\underline{\text{mod}}\ \Lambda$ ($\overline{\text{mod}}\ \Lambda$) is the group $\text{Hom}_\Lambda(A,B)$ modulo the subgroups consisting of those Λ-morphisms from A to B which factor through projective objects in mod Λ (which factor through injective objects in mod Λ). We denote this factor group by $\underline{\text{Hom}}_\Lambda(A,B)$ (by $\overline{\text{Hom}}_\Lambda(A,B)$) and denote by $\underline{f}$ ($\overline{f}$), the image in $\underline{\text{Hom}}_\Lambda(A,B)$ ($\overline{\text{Hom}}_\Lambda(A,B)$) of a morphism f in $\text{Hom}_\Lambda(A,B)$. The composition in $\underline{\text{mod}}\ \Lambda$ ($\overline{\text{mod}}\ \Lambda$) is given by $\underline{g}\ \underline{f} = \underline{gf}$ ($\overline{g}\ \overline{f} = \overline{gf}$). If we let $\text{mod}_P\Lambda$ ($\text{mod}_I\Lambda$) be the full subcategory of mod Λ consisting of those Λ-modules with no non-zero projective summands (no non-zero injective summands), then the full subcategory $\underline{\text{mod}}_P\Lambda$ ($\overline{\text{mod}}_I\Lambda$) of $\underline{\text{mod}}\ \Lambda$ ($\overline{\text{mod}}\ \Lambda$) consisting of the objects in $\text{mod}_P\Lambda$ ($\text{mod}_I\Lambda$) is dense in $\underline{\text{mod}}\ \Lambda$ ($\overline{\text{mod}}\ \Lambda$), i.e. every object in $\underline{\text{mod}}\ \Lambda$ ($\overline{\text{mod}}\ \Lambda$) is isomorphic to something in $\underline{\text{mod}}_P\Lambda$ ($\overline{\text{mod}}_I(\Lambda)$). Finally, the duality $D\colon \text{mod}\ \Lambda \longrightarrow \text{mod}\ \Lambda^{op}$ induces a duality $\text{mod}_P\Lambda \longrightarrow \text{mod}_I\Lambda^{op}$ which in turn induces a duality $\underline{\text{mod}}_P\Lambda \longrightarrow \overline{\text{mod}}_I\Lambda^{op}$.

We now recall the duality $\text{Tr}\colon \underline{\text{mod}}_P\Lambda \longrightarrow \underline{\text{mod}}_P\Lambda^{op}$ which plays an important role in our discussion. Let M be in $\text{mod}_P\Lambda$ and let $P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$ be a minimal projective presentation for M. Then define TrM to be the $\text{Coker}(P_0^* \longrightarrow P_1^*)$ where $X^* = \text{Hom}_\Lambda(X,\Lambda)$. Clearly TrM is in $\text{mod}_P\Lambda^{op}$. While this map on objects can not be extended to a functor from $\text{mod}_P\Lambda$ to $\text{mod}_P\Lambda^{op}$, it can be extended to a functor $\text{Tr}\colon \underline{\text{mod}}_P\Lambda \longrightarrow \underline{\text{mod}}_P\Lambda^{op}$ which is called the transpose and is easily seen to be a duality. Finally the composite functors

$$\underline{\text{mod}}_P\Lambda \xrightarrow{\text{Tr}} \underline{\text{mod}}_P\Lambda^{op} \xrightarrow{D} \overline{\text{mod}}_I\Lambda$$

$$\overline{\text{mod}}_I\Lambda \xrightarrow{D} \underline{\text{mod}}_P\Lambda^{op} \xrightarrow{\text{Tr}} \underline{\text{mod}}_P\Lambda$$

are equivalences of categories which are inverses of each other. In particular we have the following

Proposition 0: For an M in $\text{mod}_P\Lambda$ the following are equivalent:

i) M is indecomposable

ii) TrM is indecomposable in $\text{mod}_P\Lambda^{op}$

iii) $DTrM$ is indecomposable in $\text{mod}_I\Lambda$

iv) $\exists$ a unique indecomposable Y in $\text{mod}_I\Lambda$ such that $M \cong TrD(Y)$, namely $Y \cong TrDM$

Clearly the operation of End M on M induces an $(\text{End } M)^{op}$-module structure on $\text{Ext}^1_\Lambda(M,DTrM)$. Since the endomorphisms of M which factor through projectives operate as 0 on $\text{Ext}^1_\Lambda(M,DTrM)$, we can consider $\text{Ext}^1_\Lambda(M,DTrM)$ as a module over $\underline{\text{End}}\ M^{op}$. This $\underline{\text{End}}\ M^{op}$-module has the following properties:

Proposition 1.

a) $\text{Ext}^1_\Lambda(M,DTrM) \cong D(\underline{\text{End}}\ M)$. Hence $\text{Ext}^1_\Lambda(M,DTrM) \cong I(\underline{\text{End}}\ M^{op}/\text{rad } \underline{\text{End}}\ M^{op})$ and is thus an injective cogenerator for $\text{Mod}(\underline{\text{End}}\ M^{op})$.

b) If $f\colon DTrM \longrightarrow X$ is a morphism in $\text{Mod }\Lambda$, then $\text{Ext}^1_\Lambda(M,f)\colon \text{Ext}^1_\Lambda(M,DTrM) \longrightarrow \text{Ext}^1_\Lambda(M,X)$ is a monomorphism if and only if f is a splitable monomorphism, i.e. if and only if there is a $g\colon X \longrightarrow DTrM$ such that $gf = \text{id}_{DTrM}$.

One particularly significant consequence of b) is

Theorem 2. Let X be in $\text{mod}_I\Lambda$. Then the C-morphism

$$\text{Hom}_\Lambda(X,DTrM) \longrightarrow \text{Hom}_{\underline{\text{End}}(M)^{op}}(\text{Ext}^1_\Lambda(M,X),\text{Ext}^1_\Lambda(M,DTrM))$$

given by $f \longmapsto \text{Ext}^1_\Lambda(M,f)$ is an isomorphism while $\text{Hom}_{\underline{\text{End}}(M)^{op}}(\text{Ext}^1_\Lambda(M,X),\text{Ext}^1_\Lambda(M,DTrM)) \cong \text{Hom}_C(\text{Ext}^1_\Lambda(M,X),I(C/\text{rad } C))$ as C-modules.

Theorem 2 is particularly useful in studying the functor $F\colon \text{Mod }\Lambda \longrightarrow \text{Mod } \underline{\text{End}}(M)^{op}$ given by $F(X) = \text{Ext}^1_\Lambda(M,X)$ for all X in $\text{Mod }\Lambda$. We now give some results concerning this functor.

Proposition 3. Let M be in $\text{mod}_P\Lambda$. Then the functor $F\colon \text{Mod}\ \Lambda \longrightarrow \text{Mod}\ \underline{\text{End}}\ M^{op}$ given by $F(X) = \text{Ext}^1(M,X)$ has the following properties:

a) If X is a finitely generated Λ-module, then $F(X)$ is a finitely generated $\underline{\text{End}}\ M^{op}$-module.

b) If $\text{Ext}^1_\Lambda(M,\Lambda) = 0$, then there is a finitely generated Λ-module X such that $F(X)$ is a generator for $\text{Mod}\ \underline{\text{End}}\ M^{op}$.

c) If $\text{Ext}^1_\Lambda(M,M) = 0 = \text{Ext}^1_\Lambda(M,\Lambda)$, then given any $\underline{\text{End}}\ M^{op}$-module Y, there is a Λ-module X such that $F(X) \cong Y$. If Y is finitely generated $\underline{\text{End}}\ M^{op}$-module then X can be chosen to be finitely generated Λ-module. If Y is finitely generated and indecomposable, then X can be chosen to be finitely generated and indecomposable.

As an immediate consequence of b) we have

Proposition 4. Suppose $\text{Ext}^1_\Lambda(M,M) = 0 = \text{Ext}^1(M,\Lambda)$. Let $\mathcal{J}$ be the subcategory of $\text{Mod}\ \Lambda$ consisting of those indecomposable X such that $F(X)$ is indecomposable and let $\mathcal{S}$ be the subcategory of indecomposable modules in $\text{mod}\ \underline{\text{End}}\ M^{op}$. Then the functor $F\colon \mathcal{J} \longrightarrow \mathcal{S}$ is dense. Thus the cardinality of the isomorphism classes of objects in $\mathcal{S}$ is at most the cardinality of the isomorphism classes of objects in $\mathcal{J}$. In particular if Λ is of finite representation type so is $\underline{\text{End}}\ M^{op}$.

Note: There are examples of indecomposable Λ-modules M satisfying $\text{Ext}^1_\Lambda(M,M) = 0 = \text{Ext}^1_\Lambda(M,\Lambda)$ with Λ of finite representation type such that $\text{End}\ M^{op}$ is not of finite representation type. It would be interesting to know if Proposition 4 can be used to give new examples of rings of finite representation type.

We now turn our attention to almost split sequences. Suppose M in $\text{mod}_P\Lambda$ is indecomposable. Then $\text{End}\ M^{op}$, and hence $\underline{\text{End}}\ M^{op}$, is a local ring. Since by Propositi[on 2] we know that the $\underline{\text{End}}\ M^{op}$-modules $\text{Ext}^1_\Lambda(M,DTrM)$ and $I(\underline{\text{End}}\ M^{op}/\text{rad}\ \underline{\text{End}}\ M^{op})$ are isomorphic it follows that $\text{Ext}^1_\Lambda(M,DTrM)$ has a simple socle. Then, on the basis of Proposition 1, we have

Proposition 4. For a nontrivial element $0 \longrightarrow DTrM \longrightarrow V \longrightarrow M \longrightarrow 0$ of $\mathrm{Ext}^1_\Lambda(M,DTrM)$, the following properties are equivalent:

a) $0 \longrightarrow DTrM \longrightarrow V \longrightarrow M \longrightarrow 0$ generates the socle of $\mathrm{Ext}^1_\Lambda(M,DTrM)$.

b) If Y is an arbitrary Λ-module and $g: DTrM \longrightarrow Y$ is not a splitable monomorphism then there is an $h: V \longrightarrow Y$ such that $g = hi$.

c) If X is an arbitrary Λ-module and $f: X \longrightarrow M$ is not a splitable epimorphism, then there is an $h: X \longrightarrow V$ such that $ph = f$.

More generally we have

Proposition 5. Let $0 \longrightarrow A \xrightarrow{i} B \xrightarrow{p} C \longrightarrow 0$ be a non-split exact sequence in mod Λ with A and C indecomposable. The following statements are equivalent:

a) There is a commutative exact diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & DTrC & \rightarrow & V & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array}$$

where $0 \longrightarrow DTrC \longrightarrow V \longrightarrow C \longrightarrow 0$ is a generator for the socle of $\mathrm{Ext}^1_\Lambda(C,DTrC)$.

b) Given any generator $0 \longrightarrow DTrC \longrightarrow V \longrightarrow C \longrightarrow 0$ of the socle of $\mathrm{Ext}^1_\Lambda(C,DTrC)$, there is a commutative exact diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & DTrC & \rightarrow & V & \longrightarrow & C & \longrightarrow & \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array}$$

c) Given any morphism $g: A \longrightarrow Y$ which is not a splitable monomorphism, then there is an $h: B \longrightarrow Y$ such that $g = hi$.

c') Same as c) except that Y is assumed to be finitely generated.

d) Given any morphism $f: X \longrightarrow C$ which is not a splitable epimorphism, then there is an $h: X \longrightarrow B$ such that $f = ph$.

d') Same as d) except X is assumed to be finitely generated.

An exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ in $\operatorname{mod} \Lambda$ is said to be an almost split sequence if it has the following properties: a) it is not a splitable exact sequence; b) A and C are indecomposable and c) it satisfies any of the equivalent properties stated in Proposition 5. On the basis of our previous remarks, it is not difficult to establish

Proposition 6. a) An exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ in $\operatorname{mod} \Lambda$ is an almost split sequence if and only if the exact sequence $0 \longrightarrow D(C) \longrightarrow D(B) \longrightarrow D(A) \longrightarrow 0$ in $\operatorname{mod} \Lambda^{op}$ is an almost split sequence.

b) Given any indecomposable A in $\operatorname{mod}_I \Lambda$, there is an almost split sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$.

c) Given any indecomposable C in $\operatorname{mod}_P \Lambda$, there is an almost split sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$.

d) For two almost split sequences $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ and $0 \longrightarrow A' \longrightarrow B' \longrightarrow C' \longrightarrow 0$, the following are equivalent:

i) The sequences are isomorphic

ii) $A \approx A'$

iii) $C \approx C'$.

The rest of this talk is devoted to giving several applications of the existence of almost split sequences in diverse settings. The first result connects finitely generated Λ-modules with large, i.e. not finitely generated, Λ-modules.

Proposition 7. For a finitely generated indecomposable Λ-module M, the following statements are equivalent:

a) $\mathrm{Hom}_{\Lambda}(M,N) \neq 0$ for an infinite number of non-isomorphic finitely generated indecomposable Λ-modules N.

b) There is a denumerably generated large indecomposable Λ-module N such that $\mathrm{Hom}_{\Lambda}(M,N) \neq 0$.

c) There is a Λ-module N not having any finitely generated summands such that $\mathrm{Hom}_{\Lambda}(M,N) \neq 0$.

As a consequence of this result we have the following

Proposition 8. For an artin algebra Λ the following statements are equivalent.

a) Λ is not of finite representation type.

b) There are large indecomposable Λ-modules.

We now give an application to finite group theory of the existence of almost split sequences.

Let G be a finite group of order n, k a field whose characteristic divides the order of G and $k[G]$, the group ring of G over k. Further assume that $0 \longrightarrow L \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow k \longrightarrow 0$ is an exact sequence of $k[G]$-modules with $P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow k \longrightarrow 0$ the beginning of a minimal projective resolution of k.

Propostion 9. For a finitely generated $k[G]$-module A we have the following:

a) $A \approx L$

b) A is indecomposable and there exists a non-trivial group extension $\{1\} \longrightarrow A \longrightarrow E \longrightarrow G \longrightarrow \{1\}$ with the property that given any non-trivial group extension $\{1\} \longrightarrow B \longrightarrow E' \longrightarrow G \longrightarrow \{1\}$ with B a $k[G]$-module, there is a commutative diagram

$$\begin{array}{ccccccccc} \{1\} & \longrightarrow & B & \longrightarrow & E' & \longrightarrow & G & \longrightarrow & \{1\} \\ & & \downarrow & & \downarrow & & \| & & \\ \{1\} & \longrightarrow & A & \longrightarrow & E & \longrightarrow & G & \longrightarrow & \{1\} \end{array}$$

c) There is a non-trivial group extension $\{1\} \longrightarrow A \longrightarrow F \longrightarrow G \longrightarrow \{1\}$ with the property that given any $k[G]$-morphism $A \longrightarrow B$ which is not a splitable monomorphism, then in the pushout diagram

$$\begin{array}{ccccccccc} \{1\} & \longrightarrow & A & \longrightarrow & F & \longrightarrow & G & \longrightarrow & \{1\} \\ & & \downarrow & & \downarrow & & \| & & \\ \{1\} & \longrightarrow & B & \longrightarrow & E' & \longrightarrow & G & \longrightarrow & \{1\} \end{array},$$

the bottom row is a trivial group extension.

Note: If a $k[G]$-module A satisfies any of the above conditions, then the group extensions described in b) and c) are isomorphic. Further it is not difficult to see that $H^2(G,L) \cong k$ and so all non-trivial extensions in $H^2(G,L)$ are isomorphic and thus satisfy conditions a) and b).

We end our examples of applications with a criterion for relative projectivity of Λ-modules. Suppose $\Gamma \subset \Lambda$ is a subalgebra of Λ. We recall that a Λ-module M is relatively projective over Γ if and only if the natural morphism of Λ-modules $f\colon \Lambda \otimes_\Gamma M \longrightarrow M$ given by $f(\lambda \otimes m) = \lambda m$ is a splitable epimorphism of Λ-modules.

Proposition 10. Suppose $\Gamma \subset \Lambda$ is a subalgebra of Λ. Let C be a non-projective Λ-module and $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ an almost split sequence. C is relatively Γ-projective if and only if the sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ does not split as a sequence of Γ-modules.

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ALMOST SPLIT SEQUENCES II

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In [1] various applications of the existence and uniqueness of almost split sequences for artin algebras were given. These results motivate trying to get some more information on what the almost split sequences look like for artin algebras. This is far from being known in general. Below we discuss different ways of obtaining such information. To illustrate how our information can be applied, we give a result about periodic modules for self-injective algebras.

We shall assume throughout this paper that Λ is an artin algebra and all our modules will be finitely generated. mod Λ will denote the category of finitely generated left Λ-modules. We recall that a non-split exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ of Λ-modules is almost split if A and C are indecomposable Λ-modules, and if $h: X \to C$ is not a splitable epimorphism, then there is a map $j: X \to B$ such that $gj = h$. We now explain the content of the paper, section by section. Proofs are for the most part omitted, and a more complete and detailed version will be published elsewhere.

In section 1 we study the almost split sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ by studying the map $g: B \to C$ or the map $f: A \to B$. This approach leads maturally to the notion of irreducible maps. We discuss these maps and their connection with almost split sequences. These maps also give rise to interesting invariants for indecomposable modules.

In section 2 we discuss a method for constructing new almost split sequences from given ones, based upon equivalences between module