TWO-DIMENSIONAL PROBLEMS IN HYDRODYNAMICS AND AERODYNAMICS

L. I. SEDOV

Academy of Sciences of the USSR, Moscow University

Translated from the Russian and edited by

C. K. Chu, H. Cohen and B. Seckler

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Preface

This book constitutes a revised and amplified version of the monograph entitled 'The Theory of Two-Dimensional Ideal Fluid Flow', which appeared in 1939.

In the ensuing period the theory of the two-dimensional flow of an incompressible fluid or gas has been enriched with the results of many investigations. The new material in this edition concerns these results and those of certain other investigations which did not appear in the original.

The theory of the unsteady motion of a thin wing in an incompressible fluid (developed in detail in Moscow as far back as 1935) now includes a discussion of the thrust efficiency of an oscillating wing. These results are due to M.V.Keldysh, M.A.Lavrent'ev, A.I.Nekrasov and M.D.Khaskind.

The chapter on cascade theory has been considerably enlarged, as have been the chapters on impulsive motion of bodies in fluids and on jet theory. In particular, Section 4 of Chapter V on cavitation is new. In that section we present several schemes of cavitational flow considered by D.A.Efros in 1943-44.

A chapter devoted to the theory of planing has been introduced. Many of the advances in this subject have been made in the papers of Soviet authors. In writing Chapter VII I was greatly assisted by M.I.Gurevich, to whom I extend my sincerest thanks.

The two-dimensional theory of flow of an incompressible fluid is to a considerable extent complete, and efforts are now mainly directed toward working out the theory of gas flows. Therefore, questions of gas dynamics are also touched upon in this book.

It is impossible to include in one book all of the many facets of two-dimensional fluid and gas flow. Here we shall give in revised form the general classical theory of motion of a rigid body within a fluid and consider a number of special problems solved by the author and his closest collaborators.

The chapters devoted to motions in a compressible medium only include the theory related to certain of our papers. The presentation of other important theoretical investigations may be found by the reader in many of the monographs and handbooks that have been published recently.

In addition to giving a rational formulation and solution of some new problems, we shall, to a great extent, develop effective methods for solving basic problems of two-dimensional hydrodynamics. In particular, we shall develop methods in various branches of the subject, based on using functions such as $\sqrt{(z-a)/(z-b)}$ to isolate singularities of solutions at the points a and b, to invert certain integrals and to obtain solutions in closed form. This method, which the author first applied in thin wing theory, was subsequently widely used in wave theory, in the theory of wings of finite span, in filtration theory and in other fields. In particular, such methods were applied with much success in elasticity theory by many authors in the school of N.I.Muskhelishvili.

The chapters on the motion of a compressible fluid are related in a systematic manner with the theory of incompressible flow. The problems are analyzed in rather great detail but represent only an insignificant portion of gas dynamics, which is undergoing intensive development today.

Along with the general theory we also give the complete solution of many particular problems which involve a great number of formulas and computations.

Moscow, January 1950 L. I. Sedov

Introduction

The theory of the two-dimensional flow of an incompressible fluid or gas is a very extensively developed branch of hydrodynamics. It is the theory with which the greatest number of specific problems have been solved. Many of the results obtained satisfactorily reflect the laws and peculiarities of fluid and gas flows. They are useful models for the hydro-aerodynamic effects actually observed in nature and made practical use of in technology. Theoretical problems concerning particular three-dimensional flows are, as a rule, very difficult, and can be solved only in exceptional cases.

The greatest progress has been made in the two-dimensional potential flow of an incompressible fluid. This is explained by the fact that the powerful methods of complex function theory are applicable in this case. In many instances the use of analytic functions enables one to obtain a complete solution in a simple, effective form. These solutions are suitable for establishing characteristic qualitative properties and quantitative formulas for general classes of incompressible fluid flows and special formulas for treating particular problems. One may certainly say that the physical essence of many basic hydro-aerodynamic phenomena has been explained mathematically by means of effective methods based on the application of the theory of functions of a complex variable.

The investigation of the motion of a gas (compressible flow) is a more difficult problem. The equations of gas dynamics in the most important cases are not directly solvable by the use of functions of a complex variable. Therefore, the results in the theory of a subsonic gas flow and, more especially, transonic flow are not too numerous and are of a restricted nature.

The first systematic application of the theory of functions of a complex variable and conformal mapping in hydrodynamics was made by Helmholtz and Kirchhoff. As is well known, they considered basic problems in the theory of incompressible non-viscous jet flows.

At the end of the last century the methods for solving problems in jet theory were perfected in the papers of N.E. Joukowski and S.A. Chaplygin, who formulated and solved many new problems. The results of these papers were subsequently developed and extended by investigators in hydrodynamics of the Moscow school. An important step in jet theory was made by A.I. Nekrasov in 1922.

The intensive study carried on and the important results obtained by the Moscow school in the development of the theory of incompressible jets laid the groundwork for the appearance in 1902 of the remarkable work of Chaplygin on the theory of gas jets. This work, whose importance was underestimated in its own time, had an important and far-reaching effect on the growth of science, and is the basis of modern gas dynamics.

The advent of aviation marked the origination and rapid development of the new science of aerodynamics. Problems on the motion of a body in a fluid or air, and, specifically, the problems of the hydro-aerodynamic forces on bodies, were treated extensively.

Theoretical aerodynamics originated in the papers of Joukowski and Chaplygin, who studied the motion of a wing in two-dimensional flow and obtained all of the basic results in this field. The main difficulties were connected with the clarification of the nature of the aerodynamic forces. According to classical hydrodynamics, well developed at that time, the familiar d'Alembert paradox results in zero drag and lift in an ideal fluid. Joukowski and Chaplygin were the first to understand the true significance of d'Alembert's paradox and explained the origin of lift within the framework of the theory of ideal fluids using the solutions of two-dimensional airfoil problems. According to Joukowski's famous theorem, the presence of lift depends on the circulation of velocity around a closed contour encircling the airfoil. The second fundamental result also due to Joukowski and Chaplygin is the rule for determining the circulation. The mathematical solution of the problem of the flow past a profile allows the circulation to be arbitrary. To choose a definite value for the circulation, Joukowski and Chaplygin showed that it is necessary to satisfy the physical condition that the velocity be finite at the trailing edge of the wing. These two basic results formed the foundation for the entire subsequent development of aerodynamics.

The work of Joukowski and Chaplygin on two-dimensional problems in hydrodynamics has been extended and further developed by the Moscow school. In this connection, a large role has been played by M.V. Keldysh, M.A. Lavrent'ev and V. V. Golubev, who have applied the modern methods of complex function theory and who have solved many new important problems.

Preface to the English Edition

In our times the application of high-speed computing machines greatly extends the theoretical possibilities of approximating concrete problems. Not only do these machines provide the means for the rapid production of a massive volume of calculations and for the numerical processing of information, but they also allow for the discovery of special behavior in the solutions under study.

Along with this development of machine methods, however, analytical methods have certainly maintained their important role. There has been, in fact, an evident increase in the attention given to analytical formulation and solution. The continued importance of analytical developments can be seen in the introduction of new theoretical concepts and models, in the establishing of the essential mechanisms in various processes, in schematizing and producing new problems, in the production of new special mathematical techniques and in the working out of the necessary algorithms.

For the reasonable interpretation and presentation of results of the numerous machine calculations, and for the explanation of the various mathematical difficulties and effects encountered in the computing processes, a close connection must be maintained with deep analytical investigations of the mathematical nature and asymptotic properties of the problem being studied.

From the contents of this monograph, and from many other sources, it follows that the main body of classical results in hydrodynamics and aerodynamics have been obtained through interrelating experiments with theoretical anaylsis. This is true for qualitative and quantitative results in general problems or special phenomena. These results and conclusions have been widely used, and are of basic importance in actual practice: e.g., for the explanation and processing of experimental data, for the design of the experiments to give the data needed for direct application, and for different engineering computations and solutions of many technical questions.

The present book also develops methods for working out effective ways of solving problems and of formulating problems. These methods are based on modifying the form of given experimental laws (for example, the adiabatic equation for gases, etc.) within experimental accuracy. Such methods could be of use both in obtaining analytical solutions with the help of formulas and in suggesting or simplifying algorithms intended for machine computations. The representation of solutions with the aid of simple formulas containing prescribed defining parameters is very valuable in many practical and theoretical questions.

It is clear that in the future effective formal methods of solving problems such as those discussed in this book will still be very fruitful and useful.

The author believes that the publication of the monograph in English will promote a wider dissemination of the theories developed in two-dimensional problems in hydrodynamics and will serve to systematize the effective methods in the various applications of the theory of functions of a complex variable to hydrodynamics.

The present translation contains corrections to some errors in formulas and misprints in the Russian text.

The author thanks C. K. Chu, H. Cohen, B. D. Seckler and the publishers, John Wiley & Sons, for their efforts in producing this translation.

Moscow, November 7, 1963 L. I. Sedov

Translators' Preface

A review of articles in the Soviet technical journals which concern themselves with theoretical fluid dynamics shows frequent reference to the work 'Two-Dimensional Problems in Hydrodynamics and Aerodynamics' by L. I. Sedov. Professor Sedov's strong influence on theoretical mechanics and applied mathematics in the Soviet Union is in no small measure due to this book. In turn, the book's influence is related to the intimate correspondence which is maintained between mathematical methods and physical problems. The problems themselves are, in general, on a level above the usual graduate lectures on non-viscous fluid flows. The topics chosen are those in which function theoretic methods have proved themselves to be not only elegant but practical. The systematic employment of singular integral equations and their associated analytic function techniques seems now to be a standard operating tool of Soviet applied mathematicians. The work of Sedov and his colleagues of the Moscow School—Keldysh, Lavrentiev, Haskind—has done for fluid mechanics what Muskhelishvili and the Georgian school have done for solid mechanics.

It is therefore a great pleasure for us to be able to present Professor Sedov's book in a translation which will make it available to a wider audience. We believe that both the particular topics in fluid mechanics that are discussed and the mathematical techniques will be of value. We have tried to make this really a translation—not an interpretation. Only very rarely have we interjected a note which we felt might be of special interest to the English reader. A few mistakes have been corrected in the course of the translating and editing. We have not attempted to restyle Professor Sedov's own manner of writing. In this way, we hope that we have succeeded in producing a faithful reproduction of an excellent treatise.

The translators acknowledge the partial support of the U.S. Office of Naval Research, Fluid Dynamics Branch, and the cooperation of Mr. R. D. Cooper, head of the Fluid Dynamics Branch.

New York
1964

C. K. Chu
H. Cohen
B. Seckler

CHAPTER I

THE MOTION OF AN AIRFOIL WITH CONSTANT CIRCULATION

§1 The Perturbed Potential Flow of an Incompressible Fluid Exterior to a Profile

We shall consider the problem of determining the perturbed two-dimensional flow of an incompressible fluid occupying the entire portion of a plane exterior to some closed moving profile C, which in general will be changing in shape. The potential flow of a fluid is, generally speaking, unsteady, and we shall solve the problem on the assumption that the flow is continuous. We shall also assume that the only external forces are those due to pressure on the boundary of C and that the motion of the fluid dies out at points infinitely remote from C (the fluid is at rest at infinity).

For such a motion it follows from Kelvin's theorem that the circulation around any closed curve moving with the fluid is constant in time. From the further assumption that the flow is potential we find that the circulation around any closed curve encircling C once is the same and independent of time.

Let us determine the motion of a fluid, given its normal velocity on C and the value of the circulation Γ around C. Let x and y be the rectangular coordinates of a point in the plane of motion, and let us introduce the complex variable $z = x + iy$. We let $w(z) = \phi + i\psi$ be the complex potential of the required flow. On C we then have the following boundary condition:

$$\frac{\partial\psi}{\partial s} = \frac{\partial\phi}{\partial n} = v_n(s,t) \tag{1.1}$$

where s is the arc length along C, t is the time and v_n is the normal velocity.

If l denotes the length of C, we obviously have

$$v_n(s + l, t) = v_n(s, t)$$

Integrating relation (1.1), we obtain

$$\psi = f(s, t) + \text{const.} \tag{1.2}$$

The complex potential is thus determined to within an additive constant, and henceforth additive constants will therefore be ignored.

Generally speaking, $f(s,t)$ is not a single-valued function of the argument s. After one circuit of the contour C, the value of $f(s,t)$ and, hence, that of the stream function ψ is increased by the amount Q, the volumetric discharge of fluid through C in unit time:

$$Q = \int_C \frac{\partial \phi}{\partial n} ds = \int_C d\psi$$

If C is the boundary of a solid, Q is obviously zero.

From the condition at infinity it follows that the complex conjugate velocity can be expanded in a power series in the neighborhood of infinity as follows:

$$\frac{dw}{dz} = \frac{\Gamma + iQ}{2\pi i}\frac{1}{z} + \frac{c_2}{z^2} + \frac{c_3}{z^3} + \ldots \qquad (1.3)$$

Hence,

$$w(z) = \frac{\Gamma + iQ}{2\pi i} \ln z - \frac{c_2}{z} - \frac{c_3}{2z^2} - \ldots \qquad (1.4)$$

We now reduce our problem of the determination of the complex potential $w(z)$ to that of mapping the region exterior to C conformally onto the interior of a unit circle K.

Let $z = f(\zeta)$ be the particular function which maps the region occupied by the fluid in the z-plane conformally onto the interior of the unit circle K with center at the origin of the ζ-plane and such that the point $z = \infty$ goes into the point $\zeta = 0$.

The function $f(\zeta)$ can be expanded in a series of the form

$$z = f(\zeta) = \frac{k}{\zeta} + k_0 + k_1\zeta + k_2\zeta^2 + \ldots \qquad (1.5)$$

in which the function $P(\zeta) = k_0 + k_1\zeta + k_2\zeta^2 + \ldots$ is regular everywhere inside K. If the curve C changes shape, the coefficients k_i will depend on the time. However, if C is the boundary of a solid and if the (x, y)-axes are rigidly connected to C, the coefficients k_i will be independent of the time. In order for $f(\zeta)$ to be uniquely determined, it is sufficient to specify the image of one more point of C, say, the one that corresponds to the point $\zeta = 1$.

By substituting in $w(z)$ the value of z in terms of ζ, we obtain a function $w(\zeta)$ which is regular everywhere inside the circle K except at the point $\zeta = 0$. In the neighborhood of this point, $w(\zeta)$ is given by

$$w(\zeta) = -\frac{\Gamma + iQ}{2\pi i}\ln\zeta + c_1'\zeta + c_2'\zeta^2 + \ldots \qquad (1.6)$$

The function

$$w_0(\zeta) = \phi_0 + i\psi_0 = w(\zeta) + \frac{\Gamma + iQ}{2\pi i}\ln\zeta = c_1'\zeta + c_2'\zeta^2 + \ldots \qquad (1.7)$$

is regular everywhere inside K. On the unit circle where $\zeta = e^{i\sigma}$, because of relations (1.2), (1.6) and (1.7), the function $\psi_0(\sigma) = \psi(\sigma) + Q\sigma/2\pi$ is known and, moreover, is periodic in σ with a period of 2π.

The determination of $w_0(z)$ is easily reducible to the problem of obtaining a Fourier series expansion for $\psi_0(\sigma)$:

$$\psi_0(\sigma) = \sum_{n=1}^{\infty} a_n \cos n\sigma + b_n \sin n\sigma$$

The conjugate harmonic function $\phi_0(\sigma)$ will have the expansion

$$\phi_0(\sigma) = \sum_{n=1}^{\infty} (-a_n \sin n\sigma + b_n \cos n\sigma)$$

From this it follows that the coefficients c_n' are determined by the formula

$$c_n' = b_n + ia_n \tag{1.8}$$

It is also possible to obtain $w_0(\zeta)$, using Schwarz's formula,* which leads to the following expression for $w_0(\zeta)$:

$$w_0(\zeta) = \frac{i}{2\pi}\int_0^{2\pi} \psi_0(\sigma')\frac{e^{i\sigma'} + \zeta}{e^{i\sigma'} - \zeta}d\sigma'$$

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* Schwarz's formula is easily deduced, as follows. Let $\Phi(\zeta) = r + is$ be regular inside the circle $|\zeta| = 1$. By Cauchy's formula we have

$$\Phi(\zeta) = \frac{1}{2\pi i}\int_K \frac{\Phi(u)du}{u - \zeta} = \frac{1}{2\pi}\int_0^{2\pi} \frac{(r + is)e^{i\sigma}\,d\sigma}{e^{i\sigma} - \rho e^{i\theta}} \tag{a}$$

Furthermore,

$$0 = \frac{1}{2\pi i}\int_K \frac{\Phi(u)du}{u - \frac{1}{\bar{\zeta}}} = \frac{1}{2\pi}\int_0^{2\pi} \frac{(r + is)\rho e^{-i\theta}\,d\sigma}{\rho e^{-i\theta} - e^{-i\sigma}} \tag{b}$$

where $u = e^{i\sigma}$ and $\zeta = \rho e^{i\theta}$. Replacing i by $-i$ in (b), we can write

$$0 = -\frac{1}{2\pi}\int_0^{2\pi} (r - is)d\sigma + \frac{1}{2\pi}\int_0^{2\pi} \frac{(r - is)e^{i\sigma}\,d\sigma}{e^{i\sigma} - \rho e^{i\theta}} \tag{c}$$

Combining the equations (a) and (c), we obtain

$$\Phi(\zeta) = \frac{i}{2\pi}\int_0^{2\pi} s\,d\sigma + \frac{1}{2\pi}\int_0^{2\pi} r\frac{e^{i\sigma} + \rho e^{i\theta}\,d\sigma}{e^{i\sigma} - \rho e^{i\theta}} \tag{d}$$

This is Schwarz's formula, which expresses a function, regular within the circle, in terms of its real part on the circle. Similarly, we find a formula for $\Phi(\zeta)$ in terms of its imaginary part on the circle:

$$\Phi(\zeta) = \frac{1}{2\pi}\int_0^{2\pi} r\,d\sigma + \frac{i}{2\pi}\int_0^{2\pi} s\frac{e^{i\sigma} + \rho e^{i\theta}}{e^{i\sigma} - \rho e^{i\theta}}d\sigma$$

or

$$w_0(\zeta) = \frac{1}{2\pi}\int_K \psi_0(\zeta')\frac{\zeta' + \zeta}{\zeta' - \zeta}\frac{d\zeta'}{\zeta'} \tag{1.9}$$

The determination of the complex potential by way of obtaining the Fourier series for $\psi_0(\sigma)$ or by evaluation of the Schwarz integral entails very tedious and rather formidable computations even for the most simple cases.

Let us consider the two-dimensional motion of a rigid body in an incompressible fluid.

In this case the complex potential $w(\zeta)$ can be expressed simply in terms of $f(\zeta)$ when the motion of the body is pure translation. We shall also indicate one result which makes it easy to obtain effective expressions for $w(\zeta)$ when the motion of the body is rotational.

Since $Q = 0$ for a rigid wing, $\psi = \psi_0$ on the circle K. It is easy to see that the function $w_0(\zeta)$ introduced above is the complex potential for a fluid stream corresponding to a given motion of the wing with $\Gamma = 0$. Thus, $w_0(\zeta)$ yields the perturbed flow of a fluid in the absence of circulation.

Let U_0 and V_0 denote the components of the translational velocity, i.e., $q_0 = U_0 + iV_0$, and Ω the angular velocity of a system of rectangular coordinates attached to the moving wing. The following formula obviously holds for v_n:

$$v_n = U_0\frac{dy}{ds} - V_0\frac{dx}{ds} - \Omega\left(x\frac{dx}{ds} + y\frac{dy}{ds}\right)$$

Hence,

$$\psi_0 = U_0 y - V_0 x - \frac{\Omega}{2}(x^2 + y^2) \tag{1.10}$$

Let us now represent $w_0(z)$ in the form

$$w_0(z) = U_0 w_1(z) + V_0 w_2(z) + \Omega w_3(z) \tag{1.11}$$

where the functions $w_k(z) = \phi_k + i\psi_k$ $(k = 1, 2, 3)$ are regular in the region outside the wing, vanish at infinity and have imaginary parts ψ_k satisfying the conditions

$$\psi_1 = y, \qquad \psi_2 = -x, \qquad \psi_3 = -\tfrac{1}{2}(x^2 + y^2) \tag{1.12}$$

on the boundary of the wing.

The functions w_1, w_2 and w_3 are determined only by the geometrical properties of the boundary of the wing. It is easy to show that $w_1(z)$ is the complex potential for the perturbed potential flow of a fluid when the wing moves along the x-axis with unit velocity, $w_2(z)$ is the complex potential for a wing moving in the y-direction with unit velocity and $w_3(z)$ gives the perturbed potential flow of a fluid when the wing rotates about the origin with an angular velocity of one.

The complex potential $w_0(z)$ does not depend on the time explicitly. U_0, V_0 and Ω are functions of the time, and $w_0(z)$ is a linear combination of the three.

We now show that $w_1(\zeta)$ and $w_2(\zeta)$ or, more precisely, the combination $U_0w_1(\zeta) + V_0w_2(\zeta)$, can be expressed in closed form in terms of $f(\zeta)$. For, from (1.12) it follows that on the unit circle, i.e., when $\zeta = e^{i\sigma}$, $U_0w_1 + V_0w_2$ and $(U_0 - i\,V_0)f(\zeta)$ have the same imaginary parts. For $\zeta = 0$, $\bar{q}_0f(\zeta)$ has a first-order pole with the principal part $\bar{q}_0k/\zeta$. The imaginary parts of $\bar{q}_0k/\zeta$ and $-q_0\bar{k}\zeta$ are obviously the same for $\zeta = e^{i\sigma}$. Therefore,

$$U_0w_1(\zeta) + V_0w_2(\zeta) = \bar{q}_0f(\zeta) - \frac{\bar{q}_0k}{\zeta} - q_0\bar{k}\zeta \qquad (1.13)$$

or

$$U_0w_1(\zeta) + V_0w_2(\zeta) = (k_1\bar{q}_0 - \bar{k}q_0)\,\zeta + \bar{q}_0k_2\zeta^2 + \bar{q}_0k_3\zeta^3 + \dots$$

We now consider the determination of $w_3(\zeta)$, the complex potential for an airfoil rotating about the origin. We let

$$\bar{f}\left(\frac{1}{\zeta}\right) = \bar{k}\zeta + \bar{k}_0 + \bar{k}_1\frac{1}{\zeta} + \bar{k}_2\frac{1}{\zeta^2} + \dots$$

Clearly, for $\zeta = e^{i\sigma}$ we have $\bar{z} = \overline{f(\zeta)} = \bar{f}(1/\zeta)$, and therefore the last condition in (1.12) is expressible as

$$\operatorname{Im} w_3(\zeta) = -\frac{i}{2}f(\zeta)\bar{f}\left(\frac{1}{\zeta}\right) \qquad (1.14)$$

We next represent $-(i/2)f(\zeta)\bar{f}(1/\zeta)$ as the sum of two functions $f_1(\zeta)$ and $f_2(\zeta)$:

$$-\frac{i}{2}f(\zeta)\bar{f}\left(\frac{1}{\zeta}\right) = f_1(\zeta) + f_2(\zeta) \qquad (1.15)$$

where f_1 is regular for $|\zeta| < 1$ and f_2 is regular for $|\zeta| > 1$; at $|\zeta| = 1$, both functions are finite.

We now show that

$$w_3(\zeta) = 2f_1(\zeta) \qquad (1.16)$$

i.e., the solution of the above problem is reducible to the indicated decomposition.

In fact, for $|\zeta| = 1$ the following relations hold:

$$\operatorname{Re} f_1(\zeta) = -\operatorname{Re} f_2(\zeta) = -\operatorname{Re} f_2\left(\frac{1}{\zeta}\right)$$

and

$$\operatorname{Im} f_2(\zeta) = -\operatorname{Im} \bar{f}_2\left(\frac{1}{\zeta}\right)$$

The functions $f_1(\zeta)$ and $-\bar{f}_2(1/\zeta)$ are regular inside K and have the same real parts on K; therefore, for any ζ we have

$$f_1(\zeta) = -\bar{f}_2\left(\frac{1}{\zeta}\right) + m$$

where m is a pure imaginary constant. From the last two relations it follows that the real parts of $f_1(\zeta)$ and $f_2(\zeta)$ are the same when $|\zeta| = 1$.

Thus, the function $2f(\zeta) + m$ is regular for $|\zeta| < 1$, and its imaginary part is $-(i/2)f(\zeta)\bar{f}(1/\zeta)$ when $|\zeta| = 1$; therefore,

$$w_3(\zeta) = 2f_1(\zeta) - m$$

The constant m, being unessential, may be omitted.

The required decomposition is easily carried out when $f(\zeta)$ is a rational function. In this case $w_1(\zeta)$, $w_2(\zeta)$, and $w_3(\zeta)$ will obviously also be rational.

Using formulas (1.9) and (1.14), we obtain the following expression for $w_3(\zeta)$:

$$w_3(\zeta) = -\frac{1}{4\pi}\int_K f(\zeta')\bar{f}\left(\frac{1}{\zeta'}\right)\frac{\zeta' + \zeta}{\zeta' - \zeta}\frac{d\zeta'}{\zeta'}$$

and hence,

$$\frac{dw_3}{d\zeta} = -\frac{1}{2\pi}\int_K f(\zeta')\bar{f}\left(\frac{1}{\zeta'}\right)\frac{d\zeta'}{(\zeta' - \zeta)^2}$$

In particular,

$$\left(\frac{dw_3}{d\zeta}\right)_{\zeta=0} = c_1 = -\frac{1}{2\pi}\int_K f(\zeta')\bar{f}\left(\frac{1}{\zeta'}\right)\frac{d\zeta'}{\zeta'^2} \quad (1.17)$$

By use of the formulas derived above it is possible to write the following general formula for the complex potential $w(\zeta)$:

$$w(\zeta) = \bar{q}_0 f(\zeta) - \frac{\bar{q}_0 k}{\zeta} - q_0\bar{k}\zeta - \frac{\Omega}{4\pi}\int_K f(\zeta')\bar{f}\left(\frac{1}{\zeta'}\right)\frac{\zeta' + \zeta}{\zeta' - \zeta}\frac{d\zeta'}{\zeta'} - \frac{\Gamma}{2\pi i}\ln\zeta \quad (1.18)$$

Here the circulation Γ may be assigned arbitrarily. To determine its value, some additional requirement must be imposed.

Let us consider the case where the wing profile has a sharp trailing edge, i.e., the case of a knife-edged airfoil. We carry out the mapping of the region exterior to the airfoil onto the unit circle in the ζ-plane so that