The Mathematics of Equity Derivatives, Markets, Risk and Valuation

ANALYTICAL FINANCE
VOLUME I

JAN R. M. RÖMAN

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To my soulmate, supporter and love –
Jing Fang

Preface

This book is based upon lecture notes, used and developed for the course *Analytical Finance I* at Mälardalen University in Sweden. The aim is to cover the most essential elements of valuing derivatives on equity markets. This will also include the maths needed to understand the theory behind the pricing of the market instruments, that is, probability theory and stochastic processes. We will include pricing with time-discrete models and models in continuous time.

First, in Chap. 1 and 2 we give a short introduction to trading, risk and arbitrage-free pricing, which is the platform for the rest of the book. Then a number of different binomial models are discussed. Binomial models are important, not only to understand arbitrage and martingales, but also they are widely used to calculate the price and the Greeks for many types of derivative. Binomial models are used in trading software to handle and value several kinds of derivative, especially Bermudan and American type options. We also discuss how to increase accuracy when using binomial models. We continue with an introduction to numerical methods to solve partial differential equations (PDEs) and Monte Carlo simulations.

In Chap. 3, an introduction to probability theory and stochastic integration is given. Thereafter we are ready to study continuous finance and partial differential equations, which is used to model many financial derivatives. We focus on the Black–Scholes equation in particular. In the continuous time model, there are no solutions to American options, since they can be exercised during the entire lifetime of the contracts. Therefore we have no well-defined boundary condition. Since most exchange-traded options with stocks as underlying are of American type, we still need to use descrete models, such as the binomial model.

We will also discuss a number of generalizations relating to Black–Scholes, such as stochastic volatility and time-dependent parameters. We also discuss a number of analytical approximations for American options.

A short introduction to Poisson processes is also given. Then we study diffusion processes in general, martingale representation and the Girsanov theorem. Before finishing off with a general guide to pricing via Black–Scholes we also give an introduction to exotic options such as weather derivatives and volatility models.

As we will see, many kinds of financial instrument can be valued via a discounted expected payoff of a contingent claim in the future. We will denote this expectation $E[X(T)]$ where $X(T)$ is the so-called contingent claim at time T. This future value must then be discounted with a risk-free interest rate, r, to give the present value of the claim. If we use continuous compounding we can write the present value of the contingent claim as

$$X(t) = e^{-r(T-t)}E[X(T)].$$

In the equation above, T is the maturity time and t the present time.

Example: If you buy a call option on an underlying (stock) with maturity T and strike price K, you will have the right, but not the obligation, to buy the stock at time T, to the price K. If $S(t)$ represents the stock price at time t, the contingent claim can be expressed as $X(T) = \max\{S(T) - K, 0\}$. This means that the present value is given by

$$X(t) = e^{-r(T-t)}E[X(T)] = e^{-r(T-t)}E[max\{S(T) - K, 0\}].$$

The max function indicates a price of zero if $K \geq S(T)$. With this condition you can buy the underlying stock at a lower (same) price on the market, so the option is worthless.

By solving this expectation value we will see that this can be given (in continuous time) as the Black–Scholes–Merton formula. But generally we have a solution as

$$X(t) = S(0) \cdot Q_1(S(T) > K) - e^{-r(T-t)}K \cdot Q_2(S(T) > K),$$

where $Q_1(S(T) > K)$ and $Q_2(S(T) > K)$ make up the risk neutral probability for the underlying price to reach the strike price K in different "reference systems". This can be simplified to the Black–Scholes–Merton formula as

$$X(t) = S(0) \cdot N(d_1) - e^{-r(T-t)} K \cdot N(d_2).$$

Here d_1 and d_2 are given (derived) variables. $N(x)$ is the standard normal distribution with mean 0 and variance 1, so $N(d_2)$ represent the probability for the stock to reach the strike price K. The variables d_1 and d_2 will depend on the initial stock price, the strike price, interest rate, maturity time and volatility. The volatility is a measure of how much the stock price may vary in a specific period in time. Normally we use 252 days, since this is an approximation of the number of trading days in a year.

Also remark that by buying a call option (i.e., going long in the option contract), as in the example above, we do not take any risk. The reason is that we cannot lose more money than what we invested. This is because we have the right, but not the obligation, to fulfil the contract. The seller, on the other hand, takes the risk, since he/she has to sell the underlying stock at price K. So if he/she doesn't own the underlying stock he/she might have to buy the stock at a very high price and then sell it at a much lower price, the option strike price K. Therefore, a seller of a call option, who have the obligation to sell the underlying stock to the holder, takes a risky position if the stock price becomes higher than the option strike price.

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Contents

Notations

$B(t)$	The value of the money market account at time t
r	The risk-free interest rate
R	A short notation of $1 + r$
Ω	A sample space
ω_i	Outcome i from a sample space Ω
$S(t)$	Price of a security (financial instrument, equity, stock) at time t
$F(t)$	The forward price of a security (financial instrument, equity, stock) at time t
q	The risk neutral (risk-free) probability of an increase in price
p	The objective (real) probability or the risk-free probability of an decreasing price
Q	The risk neutral probability measure
P	The objective (real) probability measure
$E^Q[\cdot]$	The expectation value with respect to Q
$Var^Q[\cdot]$	The variance with respect to Q
ρ	The risk premium
$X(t)$	A stochastic value/process
I_t	The information set at time t
u	The binomial "up" factor with risk-neutral probability p_u or q
d	The binomial "down" factor with risk-neutral probability p_d or p
Z	A stochastic variable
$V(t)$	A value (process)
μ,α	The drift in a stochastic process
σ	The volatility in a stochastic process
t	Time
T	Time to maturity

K	The option strike price
λ	The market price of (volatility) risk (the sharp ratio)
C	A (call) option value
Δ	The change in the option value w.r.t. the underlying price, S
Γ	The change in the option Δ w.r.t. the underlying price, S
ν	The change in the option value w.r.t. the volatility, σ
Θ	The change in the option value w.r.t. time, t
ρ	The change in the option value w.r.t. the interest rate, r
d_1, d_2	Coefficients (variables) in the Black–Scholes model
VaR	Value-at-Risk
$\mathcal{F}$	A set or subsets to the sample space Ω
μ	A finite measure on a measurable space
$W(t)$	A Wiener process
$N[\mu, \sigma]$	A Normal distribution with mean μ and variance σ
τ	A stopping time (usually for American options)
L_t	A likelihood function of time t

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